Journal of Engineering Mathematics 27: 357–387, 1993.



# Multidimensional singular diffusion

J.R. KING
*Department of Theoretical Mechanics, University of Nottingham, Nottingham, NG7 2RD, U.K.*

Received 18 May 1992; accepted in revised form 4 February 1993

**Abstract.** We consider the nonlinear diffusion equation $u_t = \nabla(u^{-n}\nabla u)$ for dimension $N \geq 2$ in the cases $n > 1$, $N = 2$ and $n \geq 1$, $N \geq 3$ in which there are no finite mass solutions. We concentrate on the physically motivated case of a small but non-zero background concentration, using asymptotic methods to analyse the limit in which this background concentration goes to zero.

## 1. Introduction

This paper is largely concerned with the very widely studied nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (u^{-n}\nabla u)\,, \quad \boldsymbol{x} \in \mathbb{R}^N \tag{1.1}$$

subject to conditions

$$\left.\begin{array}{ll} \text{as } |\boldsymbol{x}| \to \infty & u \to \epsilon\,, \\ \text{at } t = 0 & u = I(\boldsymbol{x}) + \epsilon\,, \end{array}\right\} \tag{1.2}$$

where the constant $\epsilon$ is the background concentration and where

$$\int_{\mathbb{R}^N} I(\boldsymbol{x})\,\mathrm{d}V$$

is bounded. In the case of zero background concentration the total mass is therefore bounded; however, it is known ([1]) that (1.1) with $N \geq 2$ possesses finite mass solutions in $\mathbb{R}^N$ only for $n \leq 1$ when $N = 2$ and for $n < 1$ when $N \geq 3$. Here we discuss the cases in which $n$ lies outside these ranges and we incorporate a non-zero, but small, background concentration. We are motivated by, for example, the diffusion of impurities into semiconductors, these conditions being appropriate because the bulk semiconductor will never be completely pure. The non-existence of finite mass solution results from the singular behaviour of the diffusivity $D(u) = u^{-n}$ in the limit $u \to 0$. The introduction of a non-zero $\epsilon$ may thus also be viewed as regularising the diffusivity; introducing $c = u - \epsilon$ we are seeking finite mass solutions for $c$ corresponding to a diffusivity $D(c) = (c + \epsilon)^{-n}$. The corresponding one-dimensional problem has been discussed in [2]; here we extend these results by determining the asymptotic structure of the solution to (1.1) and (1.2) as $\epsilon \to 0^+$ for $N \geq 2$ with $n$ in the range for which there is no solution when $\epsilon = 0$. The approach we adopt is one of formal asymptotics based on the method of matched asymptotic expansions.

The equation (1.1) has a large number of applications in the relevant parameter ranges. Some are mentioned in [2]; another well-known example is that of diffusion of plasma where the case $n = 1$ can occur [3].

We shall begin with the case $N>2$, $n>1$ and then discuss the two borderline cases $N>2$, $n=1$ and $N=2$, $n>1$. For simplicity we largely restrict attention to the radially-symmetric problem

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}=\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}u^{-n}\frac{\partial u}{\partial r}\right), \\
&\text{at } r=0 \qquad r^{N-1}u^{-n}\frac{\partial u}{\partial r}=0, \\
&\text{as } r\to\infty \qquad u\to\epsilon, \\
&\text{at } t=0 \qquad u=I(r)+\epsilon,
\end{aligned}\right\} \tag{1.3}$$

though in Section 5 we summarise the corresponding results for the non-radially-symmetric problem. For physical reasons we assume that $I(r)$ has finite mass, and define

$$Q=\int_0^\infty r^{N-1}(u(r,t)-\epsilon)\,\mathrm{d}r, \tag{1.4}$$

which is then independent of $t$. It turns out that for $N>2$ another important constraint involves the first moment

$$M(t)=\int_0^\infty r(u(r,t)-\epsilon)\,\mathrm{d}r, \tag{1.5}$$

which we also assume to be bounded at $t=0$. It follows from the results of [4] that

$$\frac{\mathrm{d}M}{\mathrm{d}t}=-\frac{(N-2)}{(n-1)}[\epsilon^{1-n}-u^{1-n}(0,t)], \quad n\neq 1, \tag{1.6}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t}=-(N-2)[\ln(1/\epsilon)+\ln u(0,t)], \quad n=1. \tag{1.7}$$

For $N=2$, expressions (1.4) and (1.5) represent the same quantity and we introduce

$$L(t)=\int_0^\infty r\ln r\,(u(r,t)-\epsilon)\,\mathrm{d}r \tag{1.8}$$

which satisfies

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\frac{1}{n-1}[\epsilon^{1-n}-u^{1-n}(0,t)], \quad n\neq 1, N=2. \tag{1.9}$$

In the Appendices 1 and 2 we briefly discuss two other initial-boundary value problems for (1.1), which are also of physical relevance and for which the asymptotic structure is closely related to that which we derive for the conditions (1.2). Appendix 1 discusses diffusion in a finite domain; the one-dimensional version of this problem was analysed in [2]. This provides an alternative approach to ensuring that (1.1) possesses a solution (in this case (1.1) has finite mass solutions for any $n$) and is the appropriate problem physically when the finite size of the domain is more significant than the background concentration in controlling the rate of redistribution. Appendix 2 discusses the problem of nonlinear sorption.

## 2. $N>2$, $n>1$

In this section we discuss (1.3) in the parameter range $N>2$, $n>1$ and in the limit $\epsilon \to 0^+$. The analysis has a number of features in common with that of the one-dimensional problem discussed in [2], but there are also a number of important differences.

The redistribution occurs on a short timescale $t = \nu T$ where $\nu \ll 1$ is to be determined. Hence

$$\frac{\partial u}{\partial T} = \frac{\nu}{r^{N-1}} \frac{\partial}{\partial r}\left[r^{N-1} u^{-n} \frac{\partial u}{\partial r}\right] \tag{2.1}$$

and it turns out that the resulting asymptotic structure has four regions; a transition layer occurs around

$$r = s(T; \nu) ,$$

where $s$ has to be determined by matching, and we write

$$s(T; \nu) \sim s_0(T) \quad \text{as } \nu \to 0 .$$

The regions are as follows.

(1) $r < s_0(T)$.

This is a high concentration region;

$$u \sim I(r) \tag{2.2}$$

follows from (2.1).

(2) $r = s(T; \nu) + \nu z$.

The leading order behaviour in this region is governed by

$$-\dot{s}_0(u_0 - I(s_0)) = u_0^{-n} \frac{\partial u_0}{\partial z} , \tag{2.3}$$

with $\dot{s}_0 \equiv \mathrm{d}s_0/\mathrm{d}T$, and where we have matched with (2.2). It follows from (2.3) that

$$u_0 \sim [-(n-1)I(s_0)\dot{s}_0 z]^{-1/(n-1)} \quad \text{as } z \to +\infty ; \tag{2.4}$$

we note that $\dot{s}_0 < 0$.

(3) $r > s_0(T)$.

It follows from (2.4) that we should write

$$u = \nu^{1/(n-1)} \varphi .$$

Because we require $u \to \epsilon$ as $r \to \infty$ this implies that we should take

$$\nu = \epsilon^{n-1} .$$

Hence

$$\epsilon \frac{\partial \varphi}{\partial T} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left[r^{N-1} \varphi^{-n} \frac{\partial \varphi}{\partial r}\right] , \tag{2.5}$$

so imposing

$$\left.\begin{array}{ll}\text{as } r\to s_0 & \varphi_0\to+\infty\,,\\ \text{as } r\to\infty & \varphi_0\to 1\,,\end{array}\right\} \tag{2.6}$$

we obtain at leading order

$$\varphi_0=[1-(r/s_0)^{-(N-2)}]^{-1/(n-1)}\,. \tag{2.7}$$

Matching with (2.4) then implies that

$$s_0 I(s_0)\dot{s}_0=-\frac{(N-2)}{(n-1)}\,, \tag{2.8}$$

so that $s_0$ is determined by

$$\int_{s_0}^{\infty} rI(r)\,\mathrm{d}r=\frac{(N-2)}{(n-1)}\,T\,. \tag{2.9}$$

The expression (2.9) is easily shown to be consistent with (1.6).

(4) $r=O(\epsilon^{-1/2})$.

The algebraic decay associated with (2.7) is not consistent with the observation that the behaviour of the solution to (1.3) must be governed by linear diffusion when $u$ is close to $\epsilon$, so that a further region is needed. To get a diffusive balance it follows from (2.5) and (2.7) that we should write

$$r=\epsilon^{-1/2}R\,,\quad \varphi=1+\epsilon^{(N-2)/2}\Phi$$

and on matching with (2.7) we obtain the linear leading order problem

$$\left.\begin{array}{ll}\multicolumn{2}{l}{\dfrac{\partial\Phi_0}{\partial T}=\dfrac{1}{R^{N-1}}\dfrac{\partial}{\partial R}\left[R^{N-1}\dfrac{\partial\Phi_0}{\partial R}\right],}\\ \text{as } R\to 0^+ & \Phi_0\sim\dfrac{1}{(n-1)}[R/s_0]^{-(N-2)}\,,\\ \text{as } R\to\infty & \Phi_0\to 0\,,\\ \text{at } T=0 & \Phi_0=0\,.\end{array}\right\} \tag{2.10}$$

It follows from (2.10) that

$$\frac{\mathrm{d}}{\mathrm{d}T}\int_0^{\infty} R^{N-1}\Phi_0(R,T)\,\mathrm{d}R=\frac{(N-2)}{(n-1)}\,s_0^{N-2}(T)\,,$$

and hence from (2.8) that

$$\int_0^{\infty} R^{N-1}\Phi_0(R,T)\,\mathrm{d}R=\int_{s_0}^{\infty} r^{N-1}I(r)\,\mathrm{d}r\,,$$

which is a leading order expression for conservation of mass. Hence the mass which is lost

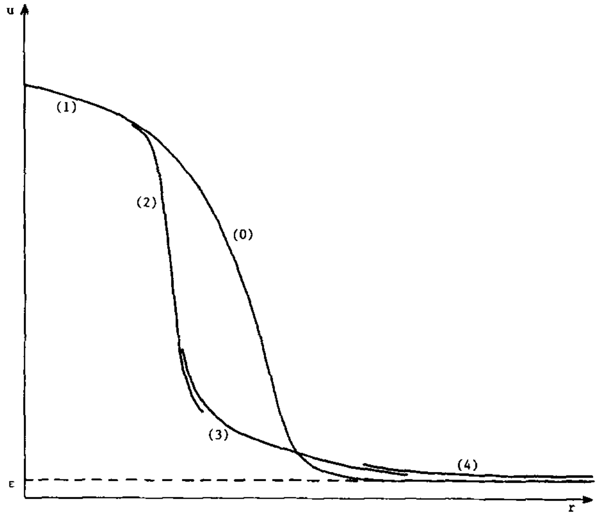


*Fig. 1.* Schematic of asymptotic structure of solution to (1.3).
$T=0$ (0) $u=I(r)$.
$0<T<T_c$ (1) $r<s_0(T)$ (2.2); (2) $r=s(T;\nu)+\nu z$ (2.3);
(3) $r>s_0(T)$ (2.7); (4) $r=\epsilon^{-1/2}R$ (2.10).

from region (1) as the front $s_0(T)$ moves inwards makes its appearance in region (4). The asymptotic structure is shown schematically in Fig. 1.

We now discuss the behaviour close to the extinction time, $T=T_c$, of the high concentration region (1), $T_c$ being defined by

$$s(T_c;\nu)=0\,,$$

with

$$s_0(T_{c0})=0\,.$$

It follows from (2.9) that

$$T_{c0}=\frac{(n-1)}{(N-2)}\int_0^\infty rI(r)\,\mathrm{d}r\,,$$

so that $T_{c0}$ depends on the initial conditions only through their first moment, and that

$$s_0\sim\left[\frac{2(N-2)}{(n-1)I(0)}(T_{c0}-T)\right]^{1/2}\quad\text{as } T\to T_{c0}^-\,. \tag{2.11}$$

It is clear from (2.3) that the lengthscale of variation in the transition region (2) is proportional to $1/\dot{s}_0$, so that the transition region becomes narrower and narrower as $T\to T_{c0}^-$. Although, in contrast to the one-dimensional case discussed in [2], it appears that the thicknesses of regions (1) and (2) do not become comparable (both are proportional to $(T_{c0}-T)^{1/2}$ as $T\to T_{c0}^-$), the asymptotic structure outlined above in fact breaks down on timescales on which $T-T_c$ is exponentially small. These intermediate asymptotic timescales are discussed in Appendix 3.

At $T = T_c$ the high concentration regions (1) and (2) disappear and the conditions (2.6) in region (3) are replaced by

$$\text{at } r = 0 \quad \frac{\partial \varphi_0}{\partial r} = 0 ,$$

$$\text{as } r \to \infty \quad \varphi_0 = 1 ,$$

so that $\varphi_0 \equiv 1$ and region (3) therefore also no longer needs to be considered. At $T = T_c$ the behaviour thus becomes linear everywhere; the full problem for region (4) reads

$$\left.\begin{aligned}
&\frac{\partial \Phi_0}{\partial T} = \frac{1}{R^{N-1}} \frac{\partial}{\partial R}\left[R^{N-1} \frac{\partial \Phi_0}{\partial R}\right] , \\
&\text{at } R = 0 , \quad T < T_{c0} \quad R^{N-1} \frac{\partial \Phi_0}{\partial R} = -\frac{(N-2)}{(n-1)} s_0^{N-2} , \\
&\qquad\qquad\quad T > T_{c0} \quad R^{N-1} \frac{\partial \Phi_0}{\partial R} = 0 , \\
&\text{as } R \to \infty \qquad\quad \Phi_0 \to 0 , \\
&\text{at } T = 0 \qquad\qquad \Phi_0 = 0 .
\end{aligned}\right\} \tag{2.12}$$

The intermediate asymptotics of the problem can be characterised by the behaviour of (2.12) close to $T = T_{c0}$. As $T \to T_{c0}^-$ it follows from (2.11) that we have

$$\text{as } R \to 0^+ \quad \Phi_0 \sim \frac{1}{(n-1)} [I(0)(n-1)R^2/2(N-2)(T_{c0} - T)]^{-(N-2)/2} \tag{2.13}$$

and the asymptotic behaviour of (2.12) as $T \to T_{c0}^-$ takes the form

$$\Phi_0 \sim -A \ln(T_{c0} - T) + AF_1[R/(T_{c0} - T)^{1/2}] \text{ for } R = O[(T_{c0} - T)^{1/2}] , \tag{2.14}$$

where the constant $A$ is determined as follows.

From (2.12) and (2.13) it is clear that $F_1(\eta)$ satisfies

$$\frac{d^2 F_1}{d\eta^2} + \left[\frac{N-1}{\eta} - \frac{1}{2}\eta\right] F_1 = 1$$

$$\text{as } \eta \to 0^+ \quad F_1 \sim \frac{1}{(n-1)A} [I(0)(n-1)\eta^2/2(N-2)]^{-(N-2)/2}$$

so that

$$\eta^{N-1} e^{-\eta^2/4} \frac{dF_1}{d\eta} = \int_0^\eta \eta^{N-1} e^{-\eta^2/4} d\eta - \left[\frac{N-2}{n-1}\right]^{N/2} \left[\frac{1}{2} I(0)\right]^{-(N-2)/2} \Big/ A . \tag{2.15}$$

We must choose $A$ so that $F_1$ does not blow up exponentially as $\eta \to +\infty$; hence

$$A = \left[\frac{N-2}{n-1}\right]^{N/2} \left[\frac{1}{2} I(0)\right]^{-(N-2)/2} \Big/ 2^{N-1} \Gamma(N/2) , \tag{2.16}$$

and

$$\eta^{N-1}\, e^{-\eta^2/4}\frac{dF_1}{d\eta} = -\int_\eta^\infty \eta^{N-1}\, e^{-\eta^2/4}\, d\eta\,, \tag{2.17}$$

from which it follows that

$$F_1(\eta) \sim -2\ln\eta \quad \text{as}\ \eta \to +\infty\,. \tag{2.18}$$

The constant of integration which arises on integrating (2.16) can only be determined by deriving more terms in the expansion as $T \to T_{c0}^-$.

It follows from (2.14) and (2.17) that at $T = T_{c0}$

$$\Phi_0 \sim -2A\ln R \quad \text{as}\ R \to 0^+ \tag{2.19}$$

with $A$ given by (2.15). From this we may deduce that as $T \to T_{c0}^+$

$$\Phi_0 \sim -A\ln(T - T_{c0}) + AF_2[R/(T - T_{c0})^{1/2}] \quad \text{for}\ R = O[(T - T_{c0})^{1/2}]\,, \tag{2.20}$$

where $F_2(\eta)$ satisfies

$$\eta^{N-1}\, e^{\eta^2/4}\frac{dF_2}{d\eta} = -\int_0^\eta \eta^{N-1}\, e^{\eta^2/4}\, d\eta\,, \tag{2.21}$$

so that

$$F_2(\eta) \sim -2\ln\eta \quad \text{as}\ \eta \to +\infty\,.$$

Finally, the asymptotic behaviour of (2.12) as $T \to +\infty$ is given in the usual way by the similarity solution

$$\Phi_0 \sim \frac{K}{T^{N/2}}\, e^{-R^2/4T} \tag{2.22}$$

where the constant $K$ is given by

$$K = Q/2^{N-1}\Gamma(N/2)\,.$$

The following comments may be made about the preceding analysis.

(1) The extinction time is given by $t = \epsilon^{n-1}T_c$, so that in the limit $\epsilon \to 0$ the high concentration regions disappear instantaneously.

(2) Although the extinction time $T_{c0}$ depends only on the first moment of the initial distribution, the details of $I(r)$ determine $s_0(T)$ through (2.9) and therefore influence the solution to (2.12). The precise form of $I(r)$ becomes irrelevant only when the linear late-stage behaviour (2.22) is established.

(3) In contrast to the one-dimensional case discussed in [2], linear diffusion plays an important role for all $T$ (see (2.12)) and takes over completely as soon as the extinction of the high concentration regions occurs.

### 3. $N > 2$, $n = 1$

The need for a separate approach when $n = 1$ is evident from, for example, expression (2.9). However, much of the analysis parallels that of the previous section.

We again write $t = \nu T$ with $\nu \ll 1$ to be determined; the four region asymptotic structure which describes the behaviour of the solution to (1.3) for small $\epsilon$ prior to the extinction of the high concentration regions goes as follows.

(1) $r < s_0(T)$.

We again have

$$u \sim I(r) .$$

(2) $r = s(T; \nu) + \nu z$.

We now have

$$-\dot{s}_0[u_0 - I(s_0)] = u_0^{-1} \frac{\partial u_0}{\partial z}$$

and the matching condition (2.4) is replaced by

$$\ln u_0 \sim I(s_0)\dot{s}_0 z \quad \text{as } z \to +\infty \tag{3.1}$$

with $\dot{s}_0 < 0$.

(3) $r > s_0(T)$.

The condition (3.1) motivates the introduction of

$$\psi = -\nu \ln u \tag{3.2}$$

and since $u \to \epsilon$ as $r \to \infty$ this requires

$$\nu = 1/\ln(1/\epsilon) .$$

We then have

$$e^{-\psi/\nu} \frac{\partial \psi}{\partial t} = \frac{\nu}{r^{N-1}} \frac{\partial}{\partial r}\left[r^{N-1} \frac{\partial \psi}{\partial r}\right], \tag{3.3}$$

and the leading order conditions are

$$\text{as } r \to s_0^+(T) \quad \psi_0 \to 0 ,$$

$$\text{as } r \to \infty \qquad \psi_0 \to 1 .$$

The left-hand side of (3.3) is exponentially small in $\nu$ so that

$$\psi_0 = 1 - (r/s_0)^{-(N-2)} , \tag{3.4}$$

and matching with (3.1) yields

$$s_0 I(s_0)\dot{s}_0 = -(N-2) , \tag{3.5}$$

so $s_0(T)$ is now given by

$$\int_{s_0}^{\infty} rI(r)\,\mathrm{d}r = (N-2)T\,, \tag{3.6}$$

which is consistent with (1.7).

(4) $r = O[\epsilon^{-1/2}\ln^{-1/2}(1/\epsilon)]$.

To obtain a diffusive balance in the far-field we must now write

$$r = \epsilon^{-1/2}\ln^{-1/2}(1/\epsilon)R\,, \quad u = \epsilon[1+\epsilon^{(N-2)/2}\ln^{N/2}(1/\epsilon)\Phi]$$

to give at leading order

$$\left.\begin{aligned}
&\frac{\partial \Phi_0}{\partial T} = \frac{1}{R^{N-1}}\frac{\partial}{\partial R}\left[R^{N-1}\frac{\partial \Phi_0}{\partial R}\right], \\
&\text{at } R=0 \qquad R^{N-1}\frac{\partial \Phi_0}{\partial R} = -(N-2)s_0^{N-2}\,, \\
&\text{as } R\to\infty \qquad \Phi_0 \to 0\,, \\
&\text{at } T=0 \qquad \Phi_0 = 0\,,
\end{aligned}\right\} \tag{3.7}$$

where we have matched with (3.4). It now follows using (3.5) that

$$\int_0^{\infty} R^{N-1}\Phi_0(R,T)\,\mathrm{d}R = \int_{s_0}^{\infty} r^{N-1}I(r)\,\mathrm{d}r\,,$$

which again expresses conservation of mass.

The leading order extinction time is now given by

$$T_{c0} = \frac{1}{(N-2)}\int_0^{\infty} rI(r)\,\mathrm{d}r\,,$$

and, because

$$s_0(T) \sim \left[\frac{2(N-2)}{I(0)}(T_{c0}-T)\right]^{1/2} \quad \text{as } T\to T_{c0}^-$$

is again proportional to $(T_{c0}-T)^{1/2}$, the structure for $T$ close to $T_{c0}$ is similar to that for $N>2$, $n>1$; the differences are briefly noted in Appendix 3.

## 4. $N=2$, $n>1$

The need for a different approach in this other borderline case is again evident from (2.9). The asymptotics of (1.3) are rather more delicate in this case; this is perhaps to be expected in view of the appearance of the $\ln r$ term in (1.8).

We again write $t=\nu T$ with $\nu \ll 1$ and obtain the following.

(1) $r < s_0(T)$.

In this region

$$u \sim I(r)\,,$$

as before.

(2) $r = s(T; \nu) + \nu z$.

We now recover (2.3), so the matching condition (2.4) again holds.

(3) $r > s_0(T)$.

We write

$$u = \nu^{1/(n-1)}\varphi$$

and obtain

$$\nu^{1/(n-1)}\frac{\partial\varphi}{\partial T} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\varphi^{-n}\frac{\partial\varphi}{\partial r}\right].$$

At leading order we have

$$\varphi_0 = [\alpha(T)\ln(r/s_0)]^{-1/(n-1)}, \tag{4.1}$$

where $\alpha(T)$ remains to be determined. In contrast to Section 2 we cannot impose the condition that $\varphi_0 \to 1$ as $r \to \infty$, and the analysis departs from that of Section 2. Introducing $R = \delta r$, where $\delta \ll 1$ also remains to be determined, it follows from (4.1) that for $R = O(1)$

$$u = [\nu/\alpha\ln(1/\delta)]^{1/(n-1)}\left[1 + O\left[\frac{1}{\ln(1/\delta)}\right]\right].$$

Since we expect this to imply that

$$u \sim \epsilon$$

we require that $\alpha$ be a constant and we may without loss of generality (by rescaling $\nu$) and for convenience set $\alpha = 2$.

Hence we require

$$\nu = 2\epsilon^{n-1}\ln(1/\delta),$$

and to obtain a balance in the diffusion equation we need

$$\delta^2 = \epsilon^n/\nu.$$

Writing $\nu = \epsilon^{n-1}\mu$ it follows that $\mu$ is given by $\mu = \ln(\mu/\epsilon)$, so that

$$\nu \sim \epsilon^{n-1}\ln(1/\epsilon),$$

and

$$\delta \sim \epsilon^{1/2}\ln^{-1/2}(1/\epsilon).$$

Matching (4.1) (with $\alpha = 2$) and (2.4) implies that

$$s_0 I(s_0)\dot{s}_0 = -\frac{2}{n-1}, \tag{4.2}$$

so $s_0$ is given by

$$\int_{s_0}^{\infty} rI(r)\,\mathrm{d}r = \frac{2}{n-1}T\,. \tag{4.3}$$

We are now in a position to discuss the final region.

(4) $r = O[\epsilon^{-1/2}\ln^{1/2}(1/\epsilon)]$.

As already indicated the appropriate rescalings are now

$$r = \epsilon^{-1/2}\ln^{1/2}(1/\epsilon)R\,,\quad u = \epsilon\left[1 + \frac{1}{\ln(1/\epsilon)}\Phi\right],$$

or, more precisely,

$$r = R/\delta\,,\quad u = \epsilon\left[1 + \frac{1}{2\ln(1/\delta)}\Phi\right],$$

yielding the leading order problem

$$\left.\begin{array}{ll} \dfrac{\partial\Phi_0}{\partial T} = \dfrac{1}{R}\dfrac{\partial}{\partial R}\left[R\dfrac{\partial\Phi_0}{\partial R}\right] & \\ \text{as } R = 0 & R\dfrac{\partial\Phi_0}{\partial R} = -\dfrac{2}{n-1}\,, \\ \text{as } R \to \infty & \Phi_0 \to 0\,, \\ \text{at } T = 0 & \Phi_0 = 0\,, \end{array}\right\} \tag{4.4}$$

where we have matched with (4.1). The conservation of mass result

$$\int_0^{\infty} R\Phi_0(R, T)\,\mathrm{d}R = \int_{s_0}^{\infty} rI(r)\,\mathrm{d}r$$

follows from (4.3) and (4.4).

We observe that in the cases discussed in Sections 2 and 3 it is possible to determine $s_0(T)$ directly from (1.6) and (1.7) without needing to do all the matching; the behaviour is thus controlled by the first moment constraint. A corresponding result for the current case is expressed by the following combination of (1.4) and (1.9):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\infty} r\ln[\epsilon^{1/2}\ln^{-1/2}(1/\epsilon)r](u(r,t) - \epsilon)\,\mathrm{d}r = \frac{1}{n-1}[\epsilon^{1-n} - u^{1-n}(0,t)]\,. \tag{4.5}$$

It can be shown that the left-hand side is given at leading order by

$$-\frac{1}{2}\epsilon^{1-n}\frac{\mathrm{d}}{\mathrm{d}T}\int_0^{s_0} rI(r)\,\mathrm{d}r$$

and (4.3) then follows.

The time of extinction of the high concentration region is given from (4.3) by

$$T_{c0} = \frac{n-1}{2}\int_0^{\infty} rI(r)\,\mathrm{d}r\,, \tag{4.6}$$

with

$$s_0(T) \sim \left[\frac{4}{(n-1)I(0)}(T_{c0} - T)\right]^{1/2} \quad T \to T_{c0}^{-},$$

and, as in Section 2, the asymptotic structure described above holds until $T$ is exponentially close to $T_c$ when the disappearance of the high concentration region occurs. The full leading order problem in $R = O(1)$ may be written in the form

$$\left.\begin{aligned}
&\frac{\partial \Phi_0}{\partial T} = \frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial \Phi_0}{\partial R}\right] \\
&\text{at } R = 0, \quad T < T_{c0} \quad R\frac{\partial \Phi_0}{\partial R} = -\frac{2}{n-1}, \\
&\qquad\qquad\quad T > T_{c0} \quad R\frac{\partial \Phi_0}{\partial R} = 0, \\
&\text{as } R \to \infty \qquad\qquad \Phi_0 \to 0, \\
&\text{at } T = 0 \qquad\qquad\ \ \Phi_0 = 0.
\end{aligned}\right\} \tag{4.7}$$

In contrast to (2.12), $s_0(T)$ does not appear in (4.7) so the solution is independent of the details of $I(r)$. The solution to (4.7) is easily derived for $R > 0$ in the form

$$\Phi_0 = \frac{2}{n-1}\int_{R/T^{1/2}}^{\infty} \frac{e^{-\eta^2/4}}{\eta}\,d\eta, \quad T < T_{c0},$$

$$\Phi_0 = \frac{2}{n-1}\int_{R/T^{1/2}}^{R/(T-T_{c0})^{1/2}} \frac{e^{-\eta^2/4}}{\eta}\,d\eta, \quad T > T_{c0}.$$

For $T < T_{c0}$ we have

$$\Phi_0 \sim -\frac{2}{n-1}\ln(R/T^{1/2}) + \kappa \quad \text{as } R \to 0^+,$$

where

$$\kappa = \frac{2}{n-1}\int_0^{\infty} \frac{(e^{-\eta^2/4} - H(1-\eta))}{\eta}\,d\eta$$

is a constant and $H$ is the Heaviside step function. For $T > T_{c0}$ we have

$$\Phi_0(0, T) = \frac{1}{n-1}\ln[T/(T - T_{c0})].$$

The behaviour for $T \sim T_{c0}$ for all $R > 0$ when $T < T_{c0}$ and for $R \gg (T - T_{c0})^{1/2}$ when $T > T_{c0}$ takes the form

$$\Phi_0 \sim \frac{2}{n-1}\int_{R/T_{c0}^{1/2}}^{\infty} \frac{e^{-\eta^2/4}}{\eta}\,d\eta,$$

while for $R = O[(T - T_{c0})^{1/2}]$, $T > T_{c0}$ we have

$$\Phi_0 \sim \frac{1}{n-1} \ln\left[\frac{T_{c0}}{T - T_{c0}}\right] + \frac{2}{n-1} \int_0^{R/(T-T_{c0})^{1/2}} \frac{(e^{-\eta^2/4} - 1)}{\eta} \, d\eta .$$

The intermediate asymptotic behaviour is thus similar to that for $N > 2$ and the main differences are again indicated in Appendix 3.

The following comments compare and contrast the results for this borderline case with those for $N > 2$ and for $N < 2$.

(1) For $N < 2$ the extinction time is determined by the total mass of the initial conditions (cf. [2]): for $N > 2$ it is determined by their first moment. For $N = 2$ the total mass and the first moment are the same quantity which, as expected, determined the extinction time (see (4.6)).

(2) As already noted, for $N > 2$ the details of the initial distribution remain relevant until the behaviour (2.22) is established. By contrast, for $N = 1$ it follows from the results of [2] that for $T > T_{c0}$ (in the notation of this paper) the leading order solution depends on the initial conditions only through their total mass. The results of this section show that the case $N = 2$ shares this characteristic of the one-dimensional case, the solution to (4.7) depending on the initial conditions only through $T_{c0}$ which is given by (4.6)

## 5. The non-radially-symmetric problem

In this section we briefly outline the appropriate generalisations of our earlier results which apply when $I(\boldsymbol{x})$ in (1.2) is not radially-symmetric. For simplicity we restrict attention to the range $N > 2$, $n > 1$. Prior to the extinction of the high concentration regions the asymptotic structure again has four regions. We now locate the interior layer (region (2)) at

$$T = \ell(\boldsymbol{x}; \epsilon) ,$$

with $l$ to be determined and with

$$\ell \sim \ell_0(\boldsymbol{x}) \quad \text{as } \epsilon \to 0 .$$

The appropriate time variable is again $T = t/\epsilon^{n-1}$. The asymptotic behaviour may now be described as follows.

(1) $T < \ell(\boldsymbol{x}; \epsilon)$

This is the immobile high concentration region in which

$$u \sim I(\boldsymbol{x}) .$$

(2) $\boldsymbol{x} = \boldsymbol{s}(T; \epsilon) - \epsilon^{n-1} \dfrac{\nabla \ell(\boldsymbol{s})}{|\nabla \ell(\boldsymbol{s})|} z ,$

where $\boldsymbol{s}$ satisfies

$$\ell(\boldsymbol{s}; \epsilon) = T$$

with

$$\ell_0(s_0) = T\,.$$

In this interior layer we have

$$u_0 - I(s_0) = |\nabla \ell_0(s_0)| u_0^{-n} \frac{\partial u_0}{\partial z}\,. \tag{5.1}$$

(3) $T > \ell(x; \epsilon)$

The appropriate variable is again $\varphi = u/\epsilon$, and we define $\psi$ by

$$\psi = \frac{1}{n-1} \varphi^{1-n}\,.$$

At leading order we now have the following moving boundary problem, which determines $\ell_0$ as well as $\psi_0$:

$$\left.\begin{array}{lll} \nabla^2 \psi_0 = 0 & \text{for } T > \ell_0(x)\,, & \\ \text{for } T = \ell_0(x) & \psi_0 = 0\,, & \nabla\psi_0 \cdot \nabla \ell_0 = -I(x)\,, \\ \text{as } |x| \to \infty & \psi_0 \to \dfrac{1}{n-1}\,, & \end{array}\right\} \tag{5.2}$$

where we have matched with (5.1). We note that the outward normal velocity of the moving boundary is $1/|\nabla \ell_0|$ and that the required initial condition on the moving boundary is that on $T = \ell_0(x)$ we have $|x| \to \infty$ as $T \to 0^+$ (assuming $I(x) > 0$ for finite $|x|$). The far-field behaviour of $\varphi_0$ takes the form

$$\varphi_0 \sim 1 + \frac{1}{(N-2)} \rho_0(T) r^{-(N-2)} \quad \text{as } r \to \infty\,, \tag{5.3}$$

where $r = |x|$ and where $\rho_0(T)$ is determined by solving (5.2). Expression (5.3) provides the matching condition for the final region.

(4) $x = \epsilon^{-1/2} X$, $\varphi = 1 + \epsilon^{(N-2)/2} \Phi$.

The leading order problem is radially symmetric and takes the form

$$\left.\begin{array}{ll} \dfrac{\partial \Phi_0}{\partial T} = \dfrac{1}{R^{N-1}} \dfrac{\partial}{\partial R}\left[R^{N-1} \dfrac{\partial \Phi_0}{\partial R}\right], & \\ \text{at } R = 0 & R^{N-1} \dfrac{\partial \Phi_0}{\partial R} = -\rho_0(T)\,, \\ \text{as } R \to \infty & \Phi_0 \to 0\,, \\ \text{at } T = 0 & \Phi_0 = 0\,, \\ \text{where } R = |X|\,. & \end{array}\right\} \tag{5.4}$$

This asymptotic structure is valid until the extinction time, $T = T_c$, of the high concentration regions. It is expected that the behaviour close to extinction will usually be radially-symmetric and will therefore be described by the results given earlier. Our radially-symmetric results are also applicable close to the extinction time in the corresponding outdiffusion problem discussed in [5].

A useful reformulation of (5.2) is the following. Introducing

$$w_0 = \int_{\ell_0(\boldsymbol{x})}^{T} \psi_0(\boldsymbol{x}, T')\,\mathrm{d}T'$$

gives

$$\left.\begin{array}{lll} \nabla^2 w_0 = I(\boldsymbol{x}) & \text{for } T > \ell_0(\boldsymbol{x})\,, & \\ \text{for } T = \ell_0(\boldsymbol{x}) & w_0 = 0\,, \quad \nabla w_0 = 0\,, & \\ \text{as } r \to \infty & w_0 \to \dfrac{T}{n-1}\,. & \end{array}\right\} \tag{5.5}$$

This formulation permits the calculation of the location, $\boldsymbol{x}_{c0}$, and time, $T_{c0}$, of extinction. It follows from (5.5) that

$$\frac{T}{(n-1)} - \frac{1}{(N-2)\omega_N} \int_{T>\ell_0(\boldsymbol{\xi})} \frac{1}{|\boldsymbol{x} - \boldsymbol{\xi}|^{N-2}} I(\boldsymbol{\xi})\,\mathrm{d}V_\xi = \begin{cases} w_0(\boldsymbol{x}, T) & \text{for } T > \ell_0(\boldsymbol{x}) \\ 0 & \text{for } T < \ell_0(\boldsymbol{x}) \end{cases} \tag{5.6}$$

where

$$\omega_N = 2\pi^{N/2}/\Gamma(N/2)$$

is the surface area of the $N$-dimensional unit sphere. This corresponds to the result

$$\frac{\mathrm{d}}{\mathrm{d}T} \int_{\boldsymbol{\xi} \in \mathbb{R}^N} \frac{1}{|\boldsymbol{x} - \boldsymbol{\xi}|^{N-2}} (u(\boldsymbol{\xi}, T) - \epsilon)\,\mathrm{d}V_\xi = -\frac{(N-2)}{(n-1)}\,\omega_N (1 - \epsilon^{n-1} u^{1-n}(\boldsymbol{x}, T))\,,$$

which is exact for (1.1) and (1.2) and holds for any $\boldsymbol{x}$. Using the conditions on $T = \ell_0(\boldsymbol{x})$ it is then clear from (5.6) that $\boldsymbol{x}_{c0}$ is given by the algebraic equations

$$\int_{\boldsymbol{\xi} \in \mathbb{R}^N} \frac{(\boldsymbol{x}_{c0} - \boldsymbol{\xi})}{|\boldsymbol{x}_{c0} - \boldsymbol{\xi}|^N} I(\boldsymbol{\xi})\,\mathrm{d}V_\xi = \boldsymbol{0}\,; \tag{5.7}$$

more precisely, $\boldsymbol{x}_{c0}$ occurs at the global maximum of

$$\int_{\boldsymbol{\xi} \in \mathbb{R}^N} \frac{1}{|\boldsymbol{x} - \boldsymbol{\xi}|^{N-2}} I(\boldsymbol{\xi})\,\mathrm{d}V_\xi$$

(if there is more than one global maximum then extinction occurs simultaneously at more than one point). It then follows that $T_{c0}$ is given by

$$T_{c0} = \frac{(n-1)}{(N-2)\omega_N} \int_{\boldsymbol{\xi} \in \mathbb{R}^N} \frac{1}{|\boldsymbol{x}_{c0} - \boldsymbol{\xi}|^{N-2}} I(\boldsymbol{\xi})\,\mathrm{d}V_\xi\,. \tag{5.8}$$

In the radially symmetric case these reduce to $\boldsymbol{x}_{c0} = \boldsymbol{0}$ and

$$T_{c0} = \frac{(n-1)}{(N-2)} \int_0^\infty r I(r)\,dr\,,$$

as before.

A further integral result, which follows from (5.2), is that

$$\omega_N \rho_0(T) = \frac{d}{dT} \int_{T > \ell_0(\xi)} I(\xi)\, dV_\xi ,$$

and the conservation of mass result

$$\omega_N \int_0^\infty R^{N-1} \Phi_0(R, T)\, dR = \int_{T > \ell_0(\xi)} I(\xi)\, dV_\xi$$

then follows from (5.4).

To summarise, the key part of the asymptotic formulation for $T < T_c$ is the moving boundary problem (5.2), which determines $\ell_0(x)$ and $\rho_0(T)$ and therefore also dictates the behaviour in the other regions. The extinction time $T_{c0}$ is determined by this moving boundary problem and can be calculated from (5.7) and (5.8). For $T > T_c$ the behaviour is again described everywhere by radially-symmetric linear diffusion; the problem (5.4), which is valid for $T < T_{c0}$, is supplemented by the condition

$$\text{at } R = 0 , \quad T > T_{c0} \quad R^{N-1} \frac{\partial \Phi_0}{\partial R} = 0 ,$$

as in (2.12)

## 6. Discussion

The main purpose of this section is to make some further comparisons with the one-dimensional results of [2]. We note the following.

(1) In one dimension (1.1) has no finite mass solutions for $n \geq 2$. For $N > 2$ there are none for $n \geq 1$. In the range $1 < n < 2/N$ for $N > 2$ there are finite mass solutions but they do not conserve mass. The behaviour in this final range is therefore different and will be discussed elsewhere.

(2) In [2] exact one-dimensional solutions were given for initial conditions of the form

$$\text{at } t = 0 \quad u = Q\delta(x) + \epsilon , \tag{6.1}$$

and it was shown in particular that for $n > 2$ the delta function persists for a finite time with diminishing magnitude. We note that delta function persistence for (1.1) appears to have been first observed in [6] where a particular initial-boundary value problem was discussed. Writing

$$u = \frac{\partial v}{\partial x}$$

the one-dimensional version of (1.1) can be transformed to

$$\frac{\partial v}{\partial t} = \left[\frac{\partial v}{\partial x}\right]^{-n} \frac{\partial^2 v}{\partial x^2} . \tag{6.2}$$

Delta function persistence for (1.1) is equivalent to the persistence of a discontinuity in the solution to (6.2) and this type of effect was noted in [7].

In the limit in which the initial condition in (1.1) becomes

$$\text{at } t = 0 \quad u = Q\delta(r)/r^{N-1} + \epsilon ,$$

this being the natural generalisation of (6.1), then for $N > 2$ no diffusion occurs; the first moment (1.5) is unbounded and the initial condition persists indefinitely. However, if for $N > 2$ we consider the singular initial condition

$$\text{at } t = 0 \quad u = M_0\delta(r)/r + \epsilon , \tag{6.3}$$

it then follows from (1.6) that for $n > 1$ the singularity at $r = 0$ again persists for a finite time with

$$\text{at } r = 0 \quad u = M_0(1 - t/\epsilon^{n-1}T_c)\delta(r)/r + \epsilon \quad \text{for } t < \epsilon^{n-1}T_c , \tag{6.4}$$

where $T_c = (n-1)M_0/(N-2)$. The initial condition (6.3) has zero mass associated with it for $N > 2$, so that

$$\text{for } r > 0 \quad u = \epsilon \quad \text{for all } T ,$$

and $u - \epsilon$ vanishes everywhere at $t = \epsilon^{n-1}T_c$.

It was shown in [2] that for the more general diffusion equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(u)\nabla u) ,$$

the criterion for delta function persistence in one dimension is that

$$\int_u^\infty u'D(u')\,\mathrm{d}u' < \infty \quad \text{for large } u . \tag{6.5}$$

In [8] the expression (6.5) is derived as the condition for the persistence of a discontinuity in the solution to

$$\frac{\partial v}{\partial t} = D\left[\frac{\partial v}{\partial x}\right]\frac{\partial^2 v}{\partial x^2} ,$$

which is a generalisation of (6.2).

The corresponding result for persistence of a singularity of the form (6.4) for $N > 2$ is that

$$\int_u^\infty D(u')\,\mathrm{d}u' < \infty \quad \text{for large } u . \tag{6.6}$$

This follows from the result that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -(N-2)(K(\epsilon) - K(u(0, t))) ,$$

where

$$K(u) = \int_u^\infty D(u')\,\mathrm{d}u' ,$$

which generalises (1.6).

For $N = 2$, $n > 1$ the analogous result concerns a singularity of the form

$$\text{at } t = 0 \quad u = M_0 \delta(r)/r \ln(1/r) + \epsilon .$$

(3) Our final comment draws together some of the results of [2] and of this paper. As already noted, in one dimension there are no finite mass solutions when $n \geq 2$; in three dimensions the corresponding criterion is that $n \geq 1$. The question of what happens in the range $1 \leq n < 2$ for initial conditions which are (in some sense) almost one-dimensional is of some interest. To illustrate this we consider the following example which, while slightly artificial, is nevertheless motivated by the diffusion of an ion-implanted impurity in a semiconductor. We consider the three-dimensional problem

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[u^{-n}\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[u^{-n}\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial z}\left[u^{-n}\frac{\partial u}{\partial z}\right], \\ &\text{as } x^2 + y^2 + z^2 \to \infty \quad u \to \epsilon , \\ &\text{at } t = 0 \quad u = I(x, y, z/\delta) + \epsilon , \end{aligned}\right\} \tag{6.7}$$

with $1 < n < 2$. We then assume that the initial conditions vary on the lengthscales $x = O(1)$, $y = O(1)$, $z = O(\delta)$ and we take $\delta \ll 1$ and $\epsilon \ll 1$.

A common simplification made in problems in which the initial conditions vary much more rapidly in one direction than in the others is that the problem can be accurately approximated by a one-dimensional one, which in this case would take the form

$$\begin{aligned} &\frac{\partial u}{\partial \hat{t}} = \frac{\partial}{\partial \hat{z}}\left[u^{-n}\frac{\partial u}{\partial \hat{z}}\right], \\ &\text{as } |\hat{z}| \to \infty \quad u \to 0 , \\ &\text{at } \hat{t} = 0 \quad u = I(x, y, \hat{z}) , \end{aligned} \tag{6.8}$$

where $\hat{z} = z/\delta$, $\hat{t} = t/\delta^2$; $x$ and $y$ appear in (6.8) only as parameters.

Our purpose here is to indicate that this simplification may not necessarily be valid, so that care must be taken in making the approximation of one-dimensional behaviour. We distinguish the following three cases.

(i) $\delta \ll \epsilon^{(n-1)/2}$

In this case the timescale $\hat{t} = O(1)$ is much shorter than $T = O(1)$ and (6.8) provides a valid approximation to (6.7) for $\hat{t} = O(1)$ wherever $u = O(1)$. On the timescale $\hat{t} = O(1)$ the motion of the moving boundary describing the erosion of the high concentration region is slow.

(ii) $\delta \gg \epsilon^{(n-1)/2}$

Now $T = O(1)$ is the shortest timescale and the high concentration region is eroded before significant one-dimensional diffusion can occur; the moving boundary discussed in Section 5 collapses at leading order from a surface $T = \ell_0(x, y, z)$ onto a curve of the form

$$z = 0 , \quad T = \ell_0(x, y)$$

but the behaviour is largely as described in Section 5 (the condition on $z = 0$, $T < \ell_0(x, y)$ is that $\psi_0 = 0$).

(iii) $\delta = O[\epsilon^{(n-1)/2}]$

In this special case the timescales $\hat{t} = O(1)$ and $T = O(1)$ are of the same order. One-dimensional diffusion in the $\hat{z}$-direction and the erosion (in the $(x, y)$ plane) of the high concentration region now occur simultaneously. We omit details of the coupling between these effects, it being more complicated than in the other cases.

We may summarise this point by noting that, for problems such as (6.7), the initial behaviour may be dominated either by one-dimensional or by three-dimensional effects, depending critically on the relationship between $\epsilon$ and $\delta$.

**Acknowledgement**

I am grateful to a referee for drawing my attention to references [6] and [8].

## Appendix 1. The finite domain case

For simplicity we again restrict attention to the range $N > 2$, $n > 1$. The problem we consider in this appendix takes the form

$$\left.\begin{array}{ll} \dfrac{\partial u}{\partial t} = \nabla \cdot (u^{-n} \nabla u) & \text{for } \boldsymbol{x} \in \Omega\,, \\ \text{for } \boldsymbol{x} \in \partial\Omega & \hat{\boldsymbol{n}} \cdot u^{-n} \nabla u = 0\,, \\ \text{at } T = 0 & u = I(\boldsymbol{x}) \end{array}\right\} \qquad (A1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded region, $\hat{\boldsymbol{n}}$ denotes the unit outward normal to $\partial\Omega$, and we take $|\boldsymbol{x}| = O(1/\epsilon)$ for $\boldsymbol{x} \in \partial\Omega$, where $\epsilon \ll 1$. This is the appropriate generalisation of the one-dimensional finite domain problem discussed in [2]. We take the limit $\epsilon \to 0$ with

$$\int_{\mathbb{R}^N} I(\boldsymbol{x})\, \mathrm{d}V$$

taken to be finite. The asymptotic structure is made up of four regions and we must again consider a short timescale $T = t/\nu$ with $\nu \ll 1$ to be determined. We take the interior layer which separates regions (1) and (3) to occur close to $T = \ell(\boldsymbol{x}; \epsilon)$ and we have the following:

(1) For $T < \ell(\boldsymbol{x}; \epsilon)$
we have

$$u \sim I(\boldsymbol{x})\,.$$

(2) $$\boldsymbol{x} = \boldsymbol{s}(T; \epsilon) - \epsilon^{N(n-1)} \frac{\nabla \ell(\boldsymbol{s})}{|\nabla \ell(\boldsymbol{s})|} z\,,$$

with

$$\ell(\boldsymbol{s}; \epsilon) = T\,.$$

We again have

$$u_0 - I(s_0) = |\nabla \ell_0(s_0)| u_0^{-n} \frac{\partial u_0}{\partial z} .$$

(3) For $T > \ell(x; \epsilon)$,
writing

$$u = \nu^{1/(n-1)} \varphi$$

and

$$\psi = \frac{1}{n-1} \varphi^{1-n}$$

gives

$$\left.\begin{array}{ll} \nabla^2 \psi_0 = 0 \quad \text{for } T > \ell_0(x) , & \\ \text{for } T = \ell_0(x) \quad \psi_0 = 0 , \quad \nabla \psi_0 \cdot \nabla \ell_0 = -I(x) , & \\ \text{as } r \to \infty \qquad \psi_0 \sim \frac{1}{n-1} \sigma_0^{-n}(T) \left[ \sigma_0(T) - \frac{(n-1)}{(N-2)} \rho_0(T) r^{-(N-2)} \right] , & \end{array}\right\} \tag{A1.2}$$

where $\sigma_0$ and $\rho_0$, as well as $\ell_0$ and $\psi_0$, are unknowns; we have

$$\varphi_0 \sim \sigma_0(T) + \frac{1}{(N-2)} \rho_0(T) r^{-(N-2)} \quad \text{as } r \to \infty .$$

(4) $x = X/\epsilon$ , $\varphi = \sigma(T; \epsilon) + \epsilon^{N-2} \Phi$,
where $\sigma(T; \epsilon) = \sigma_0(T)$. The asymptotic structure now departs from that of Section 5. In order to obtain a balance we require that $\nu$ be given by

$$\nu = \epsilon^{N(n-1)}$$

and we then have

$$\left.\begin{array}{ll} \sigma_0^n \dot{\sigma}_0 = \nabla^2 \Phi_0 , & \\ \text{as } R \to 0^+ \quad \Phi_0 \sim \frac{1}{(N-2)} \rho_0(T) R^{-(N-2)} , & \\ \text{for } |X| \in \partial\Omega \quad \hat{n} \cdot \nabla \Phi_0 = 0 ; & \end{array}\right\} \tag{A1.3}$$

equivalently we have the flux condition

$$\text{as } R = 0 \quad R^{N-1} \frac{\partial \Phi_0}{\partial R} = -\rho_0(T) .$$

The solution to (A1.3) is specified only to within an arbitrary additive function of $T$ which cannot be determined by leading order matching, but which may be absorbed into $\sigma(T; \epsilon)$ at $O(\epsilon^{N-2})$.

The problems (A1.2) and (A1.3) are coupled through the unknowns $\sigma_0(T)$ and $\rho_0(T)$, but they may be decoupled as follows. From (A1.3) we may deduce that

$$V\sigma_0^n\dot{\sigma}_0 = \omega_N \rho_0 \,, \tag{A1.4}$$

where $V$ is the volume of $\Omega$ (with respect to the variables $\boldsymbol{X}$). Using (A1.4) we may eliminate $\rho_0$ from (A1.2), which may then in principle be solved to determine $\psi_0$, $\ell_0$ and $\sigma_0$. The problem (A1.2) may then be solved to give $\Phi_0$.

The conservation of mass expression

$$V\sigma_0(T) = \int_{T>\ell_0(\boldsymbol{x})} I(\boldsymbol{x})\,\mathrm{d}V \tag{A1.5}$$

follows from (A1.2) and (A1.4). An important difference from the case discussed in Section 5 is that it does not seem possible to determine the extinction time of the high concentration regions without calculating the evolution of $\psi_0$ for all earlier $T$. An analysis similar to that of Section 5 does show, however, that the leading order location of extinction is again given by (5.7). We in general expect the solution to become radially-symmetric as the extinction time $T = T_c$ is approached, an analysis of which requires consideration of the timescale $T = T_c + O(\epsilon^{N-2})$ of which we omit details. It is clear from (A1.5) that

$$V\sigma_0(T_{c0}) = \int_{\mathbb{R}^N} I(\boldsymbol{x})\,\mathrm{d}V \,,$$

and for $T > T_c$ we have

$$u \sim \epsilon^N \sigma_0(T_{c0}) \text{ for all } \boldsymbol{x} \,.$$

We note that the extinction time $t = \epsilon^{N(n-1)}T_c$ again goes to zero as $\epsilon \to 0$; in this limit the material instantaneously diffuses out to infinity. For $\epsilon > 0$ the extent of diffusion is limited by the finite size of the domain; sufficiently close to the surface we have

$$u \sim \epsilon^N \sigma_0(T)$$

which plays the role of a (time-dependent) background concentration.

The analysis of this appendix indicates, in particular, that the numerical solution of (1.1) on artificially truncated domains may lead to spurious results.

## Appendix 2. Multidimensional indiffusion

The purpose of this appendix is to consider the problem of indiffusion (sorption) into a finite region, whereby the diffusing material enters through the surface rather than being present in some initial distribution. The relevant model is

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \nabla\cdot(D(u)\nabla u) \quad \text{for } \boldsymbol{x}\in\Omega\,, \\ &\text{for } \boldsymbol{x}\in\partial\Omega \quad u = 1\,, \\ &\text{at } t = 0 \qquad u = 0\,, \end{aligned}\right\} \tag{A2.1}$$

and we consider the case in which $D(u) = u^{-n}$. The practical importance of such problems is indicated in, for example, [9], which discusses the case in which the diffusivity $D(u)$ is an increasing function of $u$. Here we discuss the full range of $n$ in order to examine the differences between increasing ($n<0$) and decreasing ($n>0$) diffusivities.

The large time behaviour of (A2.1) is clearly given by

$$u \to 1 \quad \text{as } t \to +\infty$$

for any $D(u)$; here we consider the small time behaviour, this being the limit in which the solution depends most dramatically on the value of $n$. The cases $n<2$, $n>2$ and $n=2$ must be discussed separately; the first of these has a number of subcases. In this appendix we define $\nu$ to be the inward normal distance from a point on the boundary $\partial\Omega$.

(a) $n<2$

For small $t$ we have

$$u \sim g(\eta) \quad \text{for } \nu = O(t^{1/2}), \tag{A2.2}$$

where $\eta = \nu/t^{1/2}$ and where $g$ is given by

$$\left.\begin{aligned} &-\frac{1}{2}\eta\frac{dg}{d\eta} = \frac{d}{d\eta}\left[g^{-n}\frac{dg}{d\eta}\right], \\ &\text{at } \eta = 0 \quad g = 1, \\ &\text{as } \eta \to +\infty \quad g = 0. \end{aligned}\right\} \tag{A2.3}$$

The behaviour elsewhere depends on whether $n$ is negative or positive.

(i) $n<0$

For negative $n$ the solution has compact support, so that the solution to (A2.3) satisfies $g=0$ for $\eta \geq \eta_0$ for some finite $\eta_0$. Hence for $\nu$ sufficiently large we have

$$u = 0.$$

The limiting case discussed in [9] in which the diffusivity increases abruptly near $u=1$ corresponds to the limit $n \to -\infty$ here (see [10], [5]). Further analytical progress is possible in this case for $t=O(1)$ and such results are used in [9] to provide an upper bound on the uptake rate defined by

$$m(t) = \int_\Omega u(\mathbf{x}, t)\, dV. \tag{A2.4}$$

(ii) $0<n<2$

In this range the formulation (A2.2) holds only in the boundary layer $\eta = O(1)$. It follows from (A2.3) that

$$g \sim \left[\frac{n}{2(2-n)}\eta\right]^{-1/n} \quad \text{as } \eta \to +\infty$$

so that the behaviour elsewhere is given by a separable solution

$$u \sim t^{1/n} h(\mathbf{x}), \tag{A2.5}$$

with

$$\left.\begin{aligned} &\frac{1}{n}h = \nabla\cdot(h^{-n}\nabla h)\,, \\ &\text{as } \nu\to 0^+ \quad h\to +\infty\,; \end{aligned}\right\} \tag{A2.6}$$

more precisely we have

$$h \sim \left[\frac{n}{2(2-n)}\nu\right]^{-1/n} \quad \text{as } \nu\to 0^+\,. \tag{A2.7}$$

(iii) $n = 0$
In the case of linear diffusion we have

$$g(\eta) = \operatorname{erfc}\left[\frac{\eta}{2}\right]$$

and outside the boundary layer $u$ is exponentially small. We write

$$\ln u \sim -t^{-1}f(\boldsymbol{x}) \quad \text{as } t\to 0^+ \text{ for } \boldsymbol{x}\not\in \partial\Omega\,,$$

which gives

$$f = |\nabla f|^2\,. \tag{A2.8}$$

This first order equation is to be solved subject to the initial condition

$$\text{as } \nu\to 0^+ \quad f\sim \frac{\nu^2}{4}\,.$$

On writing $f = \frac{1}{4}F^2$ we obtain the eikonal equation

$$|\nabla F|^2 = 1\,,$$
$$\text{at } \nu = 0 \quad F = 0\,.$$

The corresponding characteristic projections will cross inside $\Omega$, so the required solution to (A2.8) will contain discontinuities in $\nabla f$. At these discontinuities a rescaling is required whereby the second spatial derivative reappears at leading order; such an analysis is straightforward and we omit it.

We now turn to the case $n < 2$ in which (A2.3) has no solution and a different approach is needed.

(b) $n < 2$

For small $t$ the expression (A2.5) again holds outside the boundary layer, with $h$ given by (A2.6). The expression (A2.7) is no longer valid however; now we have

$$h \sim [K(\boldsymbol{x})\nu]^{-1/(n-1)} \quad \text{as } \nu\to 0^+ \text{ for } \boldsymbol{x}\in\partial\Omega\,,$$

where $K$ is determined by solving (A2.6). The leading order boundary layer behaviour is now quasi-steady with

$$u \sim [1 + K(\boldsymbol{x})\nu/t^{(n-1)/n}]^{-1/(n-1)} \quad \text{for } \nu = O[t^{(n-1)/n}]\,, \quad \boldsymbol{x}\in\partial\Omega\,.$$

The limiting case $n \to +\infty$ is that in which the diffusivity decreases rapidly as $u$ increases from zero, so that a consideration of this limit provides complementary results to those of [9]. Writing

$$h = f^{-1/n}$$

we obtain at leading order as $n \to +\infty$ the linear problem

$$\nabla^2 f_0 = -1\,,$$
$$\text{for } \mathbf{x} \in \partial\Omega \quad f_0 = 0\,. \tag{A2.9}$$

Although it is valid only for $t \ll 1$, if follows from (A2.5) that for large $n$ this formulation nevertheless holds until $u$ is close to one everywhere, and it therefore provides a good description of the sorption process through most of its development.

(c) $n = 2$

Expressions (A2.5) and (A2.6) again hold, but now we have

$$h \sim 1/\nu \ln^{1/2}(1/\nu) \quad \text{as } \nu \to 0^+\,,$$

and the boundary layer is given by

$$u \sim [1 + \nu/2^{1/2} t^{1/2} \ln^{-1/2}(1/t)]^{-1} \quad \text{for } \nu = O[t^{1/2} \ln^{-1/2}(1/t)]\,.$$

We may summarise our results by providing expressions for the small time behaviour of the uptake rate (A2.4), which can also be determined from

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,\mathrm{d}S$$

with $m(0) = 0$. We have for $t \ll 1$:

($a$) $n < 2$

$$m(t) \sim \left\{ \int_0^\infty g(\eta)\,\mathrm{d}\eta\, S \right\} t^{1/2}\,,$$

which is equivalent to

$$m(t) \sim \left\{ -2 \frac{\mathrm{d}g}{\mathrm{d}\eta}(0)\, S \right\} t^{1/2}\,,$$

$S$ being the surface area of $\Omega$.

(b) $n > 2$

$$m(t) \sim \left\{ \int_\Omega h(\mathbf{x})\,\mathrm{d}V \right\} t^{1/n}\,,$$

i.e.

$$m(t) \sim \left\{ \frac{n-1}{n} \int_{\partial\Omega} K(\mathbf{x})\,\mathrm{d}S \right\} t^{1/n}\,.$$

(c) $n = 2$

$$m(t) \sim 2^{1/2} S t^{1/2} \ln^{1/2}(1/t) .$$

Limiting cases are as follows

$$n \to -\infty \quad m(t) \sim (2/-n)^{1/2} S t^{1/2} ;$$

this follows because $g \sim (1 - \eta/\eta_0)^{-1/n}$ as $n \to -\infty$ for $\eta < \eta_0$, with $\eta_0 \sim (2/-n)^{1/2}$;

$$n \to +\infty \quad m(t) \sim V t^{1/n} ,$$

where $V$ is the volume of $\Omega$. This follows because $h \sim 1$ for $f = O(1)$. It is noteworthy that the former is proportional to the surface area of $\Omega$ and the latter to its volume.

In each of the limits $n \to -\infty$ and $n \to +\infty$ the diffusivity $D(u)$ is a very rapidly varying function of $u$. Similar behaviour to that described by these two limits can occur for more general diffusivities $D(u)$. To establish a criterion for such behaviour we write $w = D(u)$ in (A2.1) to yield the equation

$$\frac{\partial w}{\partial t} = w \nabla^2 w + (1 - DD''/D'^2)|\nabla w|^2 , \tag{A2.10}$$

where $'$ denotes $\mathrm{d}/\mathrm{d}u$. If the condition

$$|1 - DD''/D'^2| \ll 1 \tag{A2.11}$$

is satisfied then the final term of (A2.10) is negligible; if, for example, we seek a separable solution

$$w \sim t^{-1} f_0(\boldsymbol{x})$$

in this limit then we recover (A2.9).

If we write

$$D(u) = A(\alpha) \exp(\Psi(u)/\alpha) \tag{A2.12}$$

for some functions $A$ and $\Psi$, where $\alpha$ is a constant, then we have

$$1 - DD''/D'^2 = -\alpha \Psi''/\Psi'^2$$

so that (A2.11) is valid if $D(u)$ takes the form (A2.12) with $\alpha$ small. This implies that $D$ is a rapidly varying function of $u$; in the case $D(u) = u^{-n}$ with $|n|$ large we can write $\alpha = 1/n$, $A = 1$ and $\Psi(u) = -\ln u$, but it is evident that the form (A2.12) is much more general. The case of a 'delta-function diffusivity' [11],

$$D(u) = \delta(u - u_c)$$

for some constant $u_c$, can be derived from (A2.12) in the limit $\alpha \to 0$ with appropriate forms for $A$ and $\Psi$.

**Appendix 3. Intermediate asymptotics**

*$N > 2$, $n > 1$*

This appendix concerns the intermediate asymptotic timescales mentioned in Section 2. These describe the behaviour close to the extinction time of the high concentration regions. Two timescales need to be discussed for $N > 2$, $n > 1$. Guided by results such as (2.14) we first introduce

$$\xi = r/(T_c - t)^{1/2}, \quad \tau = -\ln(T_c - T),$$

to give

$$\frac{\partial u}{\partial \tau} + \frac{1}{2}\xi \frac{\partial u}{\partial \xi} = \epsilon^{n-1} \frac{1}{\xi^{N-1}} \frac{\partial}{\partial \xi}\left[\xi^{N-1} u^{-n} \frac{\partial u}{\partial \xi}\right].$$

The first of the timescales is given by $\tau^* = O(1)$, where

$$\tau = \epsilon^{-(N-2)/2}\tau^* \tag{A3.1}$$

so that $T_c - T$ is exponentially small in $\epsilon$, and the asymptotic structure is made up of six regions.

(1) $\xi < \omega_0(\tau^*)$ where $\omega_0$ remains to be determined.
Here

$$u \sim I(0).$$

(2) $\xi = \omega(\tau^*; \epsilon) + \epsilon^{n-1}\zeta$ with $\omega \sim \omega_0(\tau^*)$ as $\epsilon \to 0$.
Then

$$\frac{1}{2}\omega_0[u_0 - I(0)] = u_0^{-n} \frac{\partial u_0}{\partial \zeta}$$

so that

$$u_0 \sim \left[\frac{1}{2}(n-1)I(0)\omega_0\zeta\right]^{-1/(n-1)} \quad \text{as } \zeta \to +\infty. \tag{A3.2}$$

(3) $\xi > \omega_0(\tau^*)$, $u = \epsilon\varphi$.
Here

$$\varphi_0 = B_0(\tau^*)[1 - (\xi/\omega_0)^{-(N-2)}]^{-1/(n-1)}, \tag{A3.3}$$

where $B_0$ is given by matching with (A3.2), so that

$$\omega_0^2 = \frac{2(N-2)}{(n-1)I(0)} B_0^{1-n}. \tag{A3.4}$$

(4) $\xi = \epsilon^{-1/2}\eta^*$, $\varphi = B(\tau^*; \epsilon) + \epsilon^{(N-2)/2}\Phi^*(\eta^*, \tau^*)$ with $B \sim B_0(\tau^*)$ as $\epsilon \to 0$.

Then

$$\dot{B}_0 + \frac{1}{2}\eta^* \frac{\partial \Phi_0^*}{\partial \eta^*} = \frac{1}{\eta^{*N-1}} \frac{\partial}{\partial \eta^*}\left[\eta^{*N-1} \frac{\partial \Phi_0^*}{\partial \eta^*}\right]$$

(the balance of these three terms identifies (A3.1) as the required timescale) where

$$\dot{B}_0 = \frac{\mathrm{d}B_0}{\mathrm{d}\tau^*} .$$

Hence (matching with (A3.3))

$$\eta^{*N-1} \mathrm{e}^{-B_0^n \eta^{*2}/4} \frac{\partial \Phi_0^*}{\partial \eta^*} = -\left[\frac{N-2}{n-1}\right] B_0 \omega_0^{N-2} + B_0^n \dot{B} \int_0^{\eta^*} \eta^{*N-1} \mathrm{e}^{-B_0^n \eta^{*2}/4} \mathrm{d}\eta^* ,$$

($\Phi_0^*$ is determined only up to an arbitrary additive function of $\tau^*$ which cannot be determined by matching to this order but which can be absorbed into $B$ at $O(\epsilon^{(N-2)/2})$). Since we require that $\Phi_0^*$ does not blow up exponentially as $\eta^* \to +\infty$, we therefore have

$$\left[\frac{N-2}{n-1}\right] \omega_0^{N-2} = 2^{N-1} \Gamma\left[\frac{N}{2}\right] B_0^{-1-n(N-2)/2} \dot{B}_0 \tag{A3.5}$$

and we can take

$$\Phi_0^* = \dot{B}_0 F_1[B_0^{n/2} \eta^*] ,$$

where $F_1(\eta)$ is given by (2.17).

Using (A3.4) and (A3.5) we then find that

$$\omega_0(\tau^*) = \left[\frac{2(N-2)}{(n-1)I(0)}\right]^{1/2} [1 - \tau^*/\tau_{c0}^*]^{(n-1)/(N-2)}$$

$$B_0(\tau^*) = [1 - \tau^*/\tau_{c0}^*]^{-2/(N-2)} , \tag{A3.6}$$

where $\tau_{c0}^* = 2/(N-2)A$ and $A$ is given by (2.16).

(5) $\lambda = \epsilon^{(N-2)/2} \ln \eta^*$.

Here

$$\frac{\partial \varphi_0}{\partial \tau^*} + \frac{1}{2}\frac{\partial \varphi_0}{\partial \lambda} = 0 ,$$

with

at $\lambda = 0$ $\quad \varphi_0 = B_0(\tau^*)$

at $\tau^* = 0$ $\quad \varphi_0 = 1$ ,

giving

$$\varphi_0 = \begin{cases} B_0(\tau^* - 2\lambda) & \tau^* > 2\lambda , \\ 1 & \tau^* < 2\lambda . \end{cases} \tag{A3.7}$$

(6) $R = O(1)$.

Because

$$\tau^* - 2\lambda = -2\epsilon^{(N-2)/2}\ln R\,,$$

this region separates the two ranges in (A3.7). Here we have

$$\varphi \sim 1 + \epsilon^{(N-2)/2}\Phi_0(R, T_{c0})\,,$$

where $\Phi_0$ is the solution of (2.10). Using (2.19) and (A3.6) this is easily seen to match with (A3.7).

When $\tau^* = \tau^*_{c0} + O(\epsilon^{(N-2)/2})$ it can be seen that regions (1)–(4) above merge into one, so the second timescale we need to discuss has

$$\tau = \epsilon^{-(N-2)/2}\tau^*_c(\epsilon) + \bar{\tau} \quad \text{where } \tau^*_c \sim \tau^*_{c0} \quad \text{as } \epsilon \to 0\,,$$

and we write

$$\eta^* = \epsilon^{n/2}\bar{\eta}$$

and introduce

$$\left.\begin{aligned} \bar{r} &= \bar{\eta}\,\mathrm{e}^{-\bar{\tau}/2} = \epsilon^{-(n-1)/2}\exp\left[\frac{1}{2}\epsilon^{-(N-2)/2}\tau^*_c\right] r\,, \\ \bar{T} &= -\mathrm{e}^{-\bar{\tau}} = \exp[\epsilon^{-(N-2)/2}\tau^*_c](T - T_c)\,, \end{aligned}\right\} \tag{A3.8}$$

to give

$$\left.\begin{aligned} &\frac{\partial u_0}{\partial \bar{T}} = \frac{1}{\bar{r}^{N-1}}\frac{\partial}{\partial \bar{r}}\left[\bar{r}^{N-1}u_0^{-n}\frac{\partial u_0}{\partial \bar{r}}\right], \\ &\text{at } \bar{r} = 0 \quad \bar{r}^{N-1}u_0^{-n}\frac{\partial u_0}{\partial \bar{r}} = 0\,, \\ &\text{as } \bar{r} \to +\infty \quad u_0 \sim [2\ln \bar{r}/\tau^*_{c0}]^{-2/(N-2)}\,. \end{aligned}\right\} \tag{A3.9}$$

The far-field condition in (A3.9) follows from matching with (A3.7). The initial conditions on (A3.9) hold as $\bar{T} \to -\infty$ and can be determined by matching into the timescale $\tau^* = O(1)$. It is worth stressing that the solution to (A3.9) is of infinite mass.

The behaviour of (A3.9) as $\bar{T} \to +\infty$ takes the form

$$u_0 \sim \ln^{-2/(N-2)}\bar{T}\,\tau^{*\,2/(N-2)}_{c0} + \ln^{-N/(N-2)}\bar{T}\,\tau^{*\,N/(N-2)}_{c0}F_2[\tau^{*\,n/(N-2)}_{c0}\bar{r}/\bar{T}^{1/2}\ln^{n/(N-2)}\bar{T}]$$
$$\text{for } \bar{r} = O[\bar{T}^{1/2}\ln^{n/(N-2)}\bar{T}]\,, \tag{A3.10}$$

$$u_0 \sim [2\ln \bar{r}/\tau^*_{c0}]^{-2/(N-2)} \quad \text{for} \quad \ln \bar{r} = O(\ln \bar{T}) \quad \text{with} \quad \ln \bar{r}/\ln \bar{T} \gg \frac{1}{2}\,, \tag{A3.11}$$

where $F_2(\eta)$ is given by (2.21). The expressions (A3.10) and (2.20) can be shown to match.

The significance of the timescale $\bar{T} = O(1)$ may be indicated as follows. From Section 2 we have that

$$\text{at } r = 0 \quad u \sim \begin{cases} I(0) & \text{for } T < T_c\,, \\ \epsilon & \text{for } T > T_c\,. \end{cases}$$

For $\bar{T} = O(1)$, however, the peak concentration makes a rapid transition between these two constants. The time dependence of this transition can be characterised by (A3.10), and (A3.11) indicates much of the spatial profile. The relevant intermediate asymptotic timescale, given by (A3.8), is exponentially short in $\epsilon$, in contrast to the one-dimensional case in which it is algebraically short [2].

We now indicate the main differences from the structure outlined above which occur in the intermediate asymptotic behaviour of the two borderline cases.

*$N > 2$, $n = 1$*

In this case (A3.1) is replaced by

$$\tau = \epsilon^{-(N-2)/2} \ln^{-N/2}(1/\epsilon)\tau^*$$

and much of the asymptotic structure for $\tau^* = O(1)$ is as described above. In this case the additional timescale

$$\hat{\tau} = -\ln(1 - \tau^*/\tau_c^*)/\ln(1/\epsilon) \quad \text{with } (N-2)/2 > \hat{\tau} > 0$$

also needs to be considered, where

$$\tau_c^*(\epsilon) \sim \tau_{c0}^* = 2^N \Gamma\left[\frac{N}{2}\right][I(0)/2(N-2)]^{(N-2)/2}/(N-2)^2 .$$

On this timescale we find that the equivalent to region (3) above has $\hat{\eta} = O(1)$ where

$$\xi = \epsilon^{-1/2} \ln^{-1/2}(1/\epsilon)\eta^* \quad \text{and} \quad \eta^* = \epsilon^{1/2} e^{\ln(1/\epsilon)C(\hat{\tau};\,\epsilon)/2}\hat{\eta}$$

with

$$u \sim e^{-\ln(1/\epsilon)C}[1 + \epsilon^{(N-2)/2} \ln^{N/2}(1/\epsilon)\, e^{\ln(1/\epsilon)\hat{\tau}}\hat{\Phi}] .$$

Matching gives

$$C(\hat{\tau};\epsilon) \sim 1 - \frac{2\hat{\tau}}{N-2} + \frac{1}{\ln(1/\epsilon)}\frac{N}{N-2}\ln\left[1 - \frac{2\hat{\tau}}{N-2}\right],$$

$$\omega_0^2 = \frac{2(N-2)}{I(0)}\left[1 - \frac{2\hat{\tau}}{N-2}\right], \qquad \text{(A3.12)}$$

and

$$\hat{\Phi}_0 = \frac{2}{(N-2)\tau_{c0}^*} F_1(\hat{\eta}) ,$$

where $F_1(\eta)$ is again given by (2.17).

The remaining timescale is given by $\bar{T} = O(1)$ with

$$\bar{T} = \exp[\epsilon^{-(N-2)/2} \ln^{-N/2}(1/\epsilon)\tau_c^*](T - T_c) ,$$

$$\bar{r} = \ln^{1/2}(1/\epsilon) \exp\left[\frac{1}{2}\epsilon^{-(N-2)/2} \ln^{-N/2}(1/\epsilon)\tau_c^*\right] r ,$$

and, using (A3.12), it turns out that in (A3.9) the far-field condition is replaced by

$$\text{as } \bar{r} \to +\infty \quad u_0 \sim [2 \ln \bar{r}/\tau^*_{c0}]^{-2/(N-2)} [2 \ln \ln \bar{r}/(N-2)]^{-N/(N-2)} .$$

$N = 2$, $n > 1$

The expression (A3.1) is now replaced by

$$\tau = \ln(1/\epsilon)\tau^* .$$

A further timescale also needs to be considered in this case, namely

$$\hat{\tau} = \tau^*/\ln(1/\epsilon) .$$

The equivalent to region (3) now has $\hat{\eta} = O(1)$ where

$$\xi = \epsilon^{-1/2} \ln^{1/2}(1/\epsilon)\eta^* \quad \text{and} \quad \eta^* = \epsilon^{n/2} e^{n \ln(1/\epsilon) C(\hat{\tau};\, \epsilon)/2} \hat{\eta}$$

with

$$u \sim e^{-\ln(1/\epsilon)C}[1 + \hat{\Phi}/\ln(1/\epsilon)] .$$

We find that for $\hat{\tau} = O(1)$

$$C \sim [1 - 2\hat{\tau}/(n-1)]^{1/2}$$

$$\ln \omega \sim -\frac{1}{2}(n-1)[1 - [1 - 2\hat{\tau}/(n-1)]^{1/2}] \ln(1/\epsilon)$$

and

$$\hat{\Phi}_0 = -2 \ln \hat{\eta}/(n-1)[1 - 2\hat{\tau}/(n-1)]^{1/2} .$$

The remaining timescale now has

$$\bar{T} = \exp(\ln^2(1/\epsilon)\hat{\tau}_c)(T - T_c) ,$$

$$\bar{r} = \epsilon^{-(n-1)/2} \ln^{-1/2}(1/\epsilon) \exp\left[\frac{1}{2} \ln^2(1/\epsilon)\hat{\tau}_c\right] r ,$$

where

$$\hat{\tau}_c(\epsilon) \sim \frac{1}{2}(n-1) ,$$

and in (A3.9) the far-field condition becomes

$$\text{as } \bar{r} \to +\infty \quad \ln u_0 \sim -2 \ln^{1/2} \bar{r}/(n-1)^{1/2};$$

the determination of the pre-exponential term in the far-field expression for $u_0$ requires higher order matching. The intermediate asymptotic timescale in this case is thus exponentially short in $\ln^2(1/\epsilon)$.

## References


1. J.L. Vazquez, Nonexistence of solutions for nonlinear heat equations of fast-diffusion type. Preprint.
2. J.R. King, 'Instantaneous source' solutions to a singular nonlinear diffusion equation, *J. Eng. Math.* 27 (1993) 31–72.
3. J.G. Berryman and C.J. Holland, Asymptotic behaviour of the nonlinear diffusion equation $n_t = (n^{-1}n_x)_x$. *J. Math. Phys.* 23 (1982) 983–987.
4. J.R. King, Integral results for nonlinear diffusion equations. *J. Eng. Math.* 25 (1991) 191–205.
5. J.R. King, Asymptotic results for nonlinear outdiffusion, submitted.
6. M. Bertsch and M. Ughi, Positivity properties of viscosity solutions of a degenerate parabolic equation. *Nonlinear Analysis* 14 (1990) 571–592.
7. P. Rosenau, P.S. Hagan, R.L. Northcutt and D.S. Cohen, Delayed diffusion due to flux limitation. *Phys. Lett.* A 142 (1989) 26–30.
8. M. Bertsch and R. Dal Passo, Hyperbolic phenomena in a strongly degenerate parabolic equation. *Arch. Rat. Mech. Anal.* 117 (1992) 349–387.
9. P.S. Hagan and D. Brenner, Sorption limits with increasing diffusion coefficients. *SIAM J. Appl. Math.* 48 (1988) 917–920.
10. C.M. Elliott, M.A. Herrero, J.R. King and J.R. Ockendon, The mesa problem: diffusion patterns for $u_t = \nabla \cdot (u^m \nabla u)$ as $m \to +\infty$. *IMA J. Appl. Math.* 37 (1986) 147–154.
11. J.R. Philip, Theory of infiltration. *Adv. Hydrosci.* 5 (1969) 215–296.